LESSON PLAN

Course: Grade 12 U Advanced Functions **Lesson: 2 - 4**

Unit/Chapter: Functions **Topic: Algebra of Function**

- ***note:*** **The Algebra of Functions**

We can define a new function by using algebra to combine two previous functions $f(x)$ and $g(x)$. These functions can be combined is a manner similar to the way we add, subtract, multiply and divide.

Operations on Functions

Let the domain of $f(x)$ be A and the domain of $g(x)$ be B.

Addition: $(f+g)(x) = f(x)+g(x)$ Domain $= A \cap B$

Subtraction: $(f-g)(x) = f(x)-g(x)$ Domain $= A \cap B$

Multiplication: $(fg)(x) = f(x)g(x)$ Domain $= A \cap B$

Division: $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$ Domain $= A \cap B$, $g(x) \neq 0$

Given $f(x) = x^2 - 5x - 1$ with domain $A = \{x | -4 \le x \le 1, x \in \Re\}$ and $g(x) = 2x + 3$ with domain $B = \{x | -2 \le x \le 5, x \in \Re\}$, find:

a) $(f+g)(x) = f(x)+g(x)$ domain $= A \cap B$

$= (x^2 - 5x - 1) + (2x + 3)$ $\{x | -2 \le x \le 1, x \in \Re\}$

$= x^2 - 3x + 2$

b) $(f-g)(x) = f(x)-g(x)$ domain $= A \cap B$

$= (x^2 - 5x - 1) - (2x + 3)$ $\{x | -2 \le x \le 1, x \in \Re\}$

$= x^2 - 7x - 4$

c) $(fg)(x) = f(x)g(x)$ domain $= A \cap B$

$= (x^2 - 5x - 1)(2x + 3)$ $\{x | -2 \le x \le 1, x \in \Re\}$

$= 2x^3 - 7x^2 - 17x - 3$

d) $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$ domain $= A \cap B$, $g(x) \neq 0$

$= \frac{x^2 - 5x - 1}{2x + 3}$ $\left\{ x \middle| -2 \leq x \leq 1, x \neq \frac{-3}{2}, x \in \Re \right\}$

We could also add or subtract functions graphically. (Note that multiplication and division are not defined graphically.) To complete either, we add or subtract the y – values from both functions.

(insert graph)

- ***homework assignment:*** **FM12** exercise 6.4 # 2 – 4, 7

EXERCISE 6.4

A 1. Given $f(x) = x^2 + 3x + 1$ and $g(x) = x$, both defined on R.

(a) State.

(i) $(f + g)(x)$
(ii) $(f - g)(x)$
(iii) $(g - f)(x)$
(iv) $(fg)(x)$
(v) $\left(\frac{f}{g}\right)(x)$

(b) State the domains of the functions in (a).

2. For each of the following pairs of functions f and g defined on A and B respectively, state.

(a) $(f + g)(x)$
(b) $(f - g)(x)$
(c) the domain of $f + g$

(i) $f(x) = x + 2$, $A = R$, $g(x) = x^3 - 1$, $B = \{x \mid 0 \leq x \leq 1\}$
(ii) $f(x) = 8x^7$, $A = \{x \mid 0 \leq x \leq 5\}$, $g(x) = x^7 - x$, $B = \{x \mid -4 \leq x \leq 4\}$
(iii) $f(x) = x^3 - x^2 + 2x - 1$, $A = \{x \mid 1 < x < 4\}$, $g(x) = x^3 + 1$, $B = \{x \mid 0 < x < 2\}$

3. For each of the following pairs of functions f and g defined on A and B respectively, state.

(a) $(fg)(x)$ (b) the domain of fg
(c) $\left(\frac{f}{g}\right)(x)$ (d) the domain of $\frac{f}{g}$

(i) $f(x) = x + 1$, $A = R$, $g(x) = x - 1$, $B = R$
(ii) $f(x) = x$, $A = \{x \mid -3 < x < 3\}$, $g(x) = x^2 - 4$, $B = \{x \mid -5 < x < 5\}$
(iii) $f(x) = \sqrt{x + 1}$, $A = \{x \mid x \geq -1\}$, $g(x) = \sqrt{x}$, $B = \{x \mid x \geq 0\}$

B 4. For each of the following pairs of functions f and g defined on A and B respectively, find

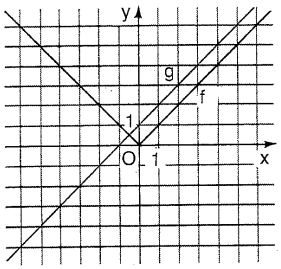
(a) $f + g$ (b) $f - g$
(c) $g - f$ (d) fg
(e) $\frac{f}{g}$

and the domains of these functions.

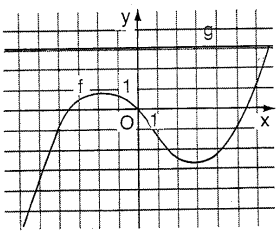
(i) $f(x) = x^2 + 2$, $A = R$, $g(x) = x^2 - 3x + 2$, $B = R$
(ii) $f(x) = x^3 - 1$, $A = R$, $g(x) = x^2 + 4$, $B = \{x \mid x < 87\}$
(iii) $f(x) = x^4 + x^2 + 1$, $A = R$, $g(x) = x^4$, $B = R$
(iv) $f(x) = \sqrt{x^2 - 1}$, $A = \{x \mid |x| \geq 1\}$, $g(x) = \sqrt{4 - x^2}$, $B = \{x \mid |x| \leq 2\}$
(v) $f(x) = x^4 - 3x^3 + x^2 - 2x - 8$, $A = R$, $g(x) = x^2 - 2x - 8$, $B = \{x \mid -2 \leq x \leq 4\}$

5. Copy each of the following graphs and use graphical addition to sketch the graph of $f + g$.

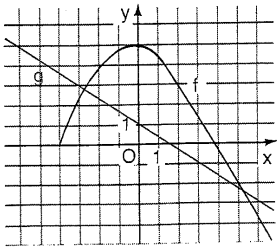
(a)



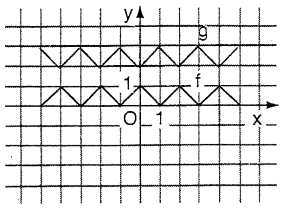
(b)



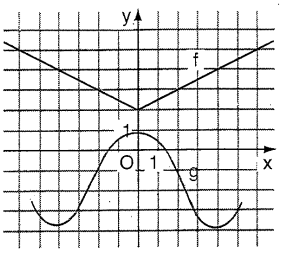
(c)



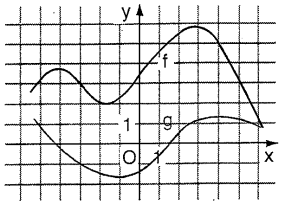
(d)



(e)



(f)



6. Copy each of the graphs in question 5 and sketch the graph of $f - g$.

7. Given the following functions with the indicated domains,
(a) graph f and g on the same axes;
(b) use graphical addition to sketch the graph of $f + g$;
(c) give a formula for $f + g$;
(d) state the domain of $f + g$.,
(i) $f(x) = 2x$, $g(x) = 1 - x$, $x \in R$
(ii) $f(x) = x^2$, $g(x) = 2x$, $x \in R$
(iii) $f(x) = x^2$ for $-2 \leq x \leq 2$, $g(x) = 1$ for $x \geq 0$
(iv) $f(x) = x$ for $x \in R$, $g(x) = \sqrt{x}$ for $x \geq 0$
(v) $f(x) = x^3$, $g(x) = -x$, $x \in R$
(vi) $f(x) = \sqrt{16 - x^2}$ for $|x| \leq 4$, $g(x) = x$ for $x \in R$

8. Given $f(x) = x^2$ and $g(x) = 2$, $-2 \leq x \leq 2$.
(a) Sketch the graphs of f and g on the same axes.
(b) Sketch the graph of $f + g$.
(c) Sketch the graph of gf.
(d) How are the graphs of $f + g$ and gf related to the graph of f?

9. If $f(x) = x^2 - 4$ and $g(x) = x$, find the values of x for which $(f + g)(x) = 2$.

10. Let $f(x) = \dfrac{1}{x^2 - 4}$ and $g(x) = \dfrac{2x}{x - 2}$.
(a) What are the domains of the functions f, g, and $f + g$?
(b) For what value of x is $(f + g)(x) = 2$?

C 11. The reciprocal function of f is the function $g(x) = \dfrac{1}{f(x)}$. Investigate the relationship between the graphs of a function and its reciprocal by studying the following special cases.
(a) Draw the graph of $f(x) = x$ and, directly beneath it, draw the graph of $g(x) = \dfrac{1}{x}$.
(b) Draw the graph of $f(x) = x^2$ and, directly beneath it, draw the graph of $g(x) = \dfrac{1}{x^2}$.
(c) (i) What happens to the graph of $g(x) = \dfrac{1}{f(x)}$ when f(x) becomes large?
(ii) What happens to the graph of $g(x) = \dfrac{1}{f(x)}$ when f(x) becomes close to 0?
(d) Use these observations to sketch the graph of the function $g(x) = \dfrac{1}{x^2 - 1}$.

MIND BENDER

Find unequal rational numbers other than 2 and 4 such that

$$a^b = b^a$$

(iii) (a) -1 (b) -1 (c) -1
(iv) (a) -1 (b) 0 (c) 1
(v) (a) 0 (b) 1 (c) 0
(vi) (a) 0 (b) 0 (c) 0

4. (a) $C(t) = \begin{cases} 12, & 0 < t \leq 3 \\ 12 + 2k, & 2 + k < t \leq 3 + k \text{ for } k = 1, 2, 3, \ldots \end{cases}$ (c) \$20

5. (a) $C(x) = \begin{cases} 1.69, & 0 < x \leq 100 \\ 1.69 + 0.34k, & 100k < x \leq 100(k + 1)\ k = 1, 2, 3, \ldots, 8, 9 \end{cases}$

6. (a) $P(t) = \begin{cases} 10t, & 0 \leq t \leq 40 \\ 400 + 15(t - 40), & t > 40 \end{cases}$ (b) \$350, \$475

EXERCISE 6.3

1. (a) domain $= \{x \mid -1 \leq x \leq 1\}$, range $= \{y \mid -2 \leq y \leq 2\}$
   (b) domain $= \{x \mid -4 \leq x \leq 1\}$, range $= \{y \mid -1 \leq y \leq 2\}$
2. (b) and (c) only
3. (a) not a function of x (b) not a function of x (c) a function of x
   (d) a function of x (e) a function of x (f) a function of x
   (g) a function of x (h) not a function of x (i) not a function of x
4. $y = \dfrac{10 + \sqrt{160 - 4x^2 - 24x}}{2}$ and $y = \dfrac{10 - \sqrt{160 - 4x^2 - 24x}}{2}$

EXERCISE 6.4

1. (a) (i) $x^2 + 4x + 1$ (ii) $x^2 + 2x + 1$ (iii) $-x^2 - 2x - 1$
   (iv) $x^3 + 3x^2 + x$ (v) $\dfrac{x^2 + 3x + 1}{x}$
   (b) (i) R (ii) R (iii) R (iv) R (v) $\{x \mid x \neq 0\}$
2. (i) (a) $x^3 + x + 1$ (b) $-x^3 + x + 3$ (c) $\{x \mid 0 \leq x \leq 1\}$
   (ii) (a) $9x^7 - x$ (b) $7x^7 + x$ (c) $\{x \mid 0 \leq x \leq 4\}$
   (iii) (a) $2x^3 - x^2 + 2x$ (b) $-x^2 + 2x - 2$ (c) $\{x \mid 1 < x < 2\}$
3. (i) (a) $x^2 - 1$ (b) R (c) $\dfrac{x + 1}{x - 1}$ (d) $\{x \mid x \neq 1\}$
   (ii) (a) $x^3 - 4x$ (b) $\{x \mid -3 < x < 3\}$ (c) $\dfrac{x}{x^2 - 4}$ (d) $\{x \mid -3 < x < 3, x \neq \pm 2\}$
   (iii) (a) $\sqrt{x^2 + x}$ (b) $\{x \mid x \geq 0\}$ (c) $\sqrt{1 + \dfrac{1}{x}}$ (d) $\{x \mid x > 0\}$
4. (i) (a) $2x^2 - 3x + 4$, R (b) $3x$, R
   (c) $-3x$, R (d) $x^4 - 3x^3 + 4x^2 - 6x + 4$, R
   (e) $\dfrac{x^2 + 2}{x^2 - 3x + 2}$, $\{x \mid x \neq 1, x \neq 2\}$
   (ii) (a) $x^3 + x^2 + 3$, $\{x \mid x < 87\}$ (b) $x^3 - x^2 - 5$, $\{x \mid x < 87\}$
   (c) $-x^3 + x^2 + 5$, $\{x \mid x < 87\}$ (d) $x^5 + 4x^3 - x^2 - 4$, $\{x \mid x < 87\}$
   (e) $\dfrac{x^3 - 1}{x^2 + 4}$, $\{x \mid x < 87\}$
   (iii) (a) $2x^4 + x^2 + 1$, R (b) $x^2 + 1$, R
   (c) $-x^2 - 1$, R (d) $x^8 + x^6 + x^4$, R
   (e) $\dfrac{x^4 + x^2 + 1}{x^4}$, $\{x \mid x \neq 0\}$
   (iv) (a) $\sqrt{x^2 - 1} + \sqrt{4 - x^2}$, $\{x \mid -2 \leq x \leq -1 \text{ or } 1 \leq x \leq 2\}$
   (b) $\sqrt{x^2 - 1} - \sqrt{4 - x^2}$, $\{x \mid -2 \leq x \leq -1 \text{ or } 1 \leq x \leq 2\}$
   (c) $\sqrt{4 - x^2} - \sqrt{x^2 - 1}$, $\{x \mid -2 \leq x \leq -1 \text{ or } 1 \leq x \leq 2\}$
   (d) $\sqrt{x^2 - 1} \times \sqrt{4 - x^2}$, $\{x \mid -2 \leq x \leq -1 \text{ or } 1 \leq x \leq 2\}$
   (e) $\dfrac{\sqrt{x^2 - 1}}{\sqrt{4 - x^2}}$, $\{x \mid -2 < x \leq -1 \text{ or } 1 \leq x < 2\}$
   (v) (a) $x^4 - 3x^3 + 2x^2 - 4x - 16$, $\{x \mid -2 \leq x \leq 4\}$

(b) $x^4 - 3x^3$, $\{x \mid -2 \leq x \leq 4\}$
(c) $-x^4 + 3x^3$, $\{x \mid -2 \leq x \leq 4\}$
(d) $x^6 - 5x^5 - x^4 + 20x^3 - 12x^2 + 32x + 64$, $\{x \mid -2 \leq x \leq 4\}$
(e) $\dfrac{x^4 - 3x^3 + x^2 - 2x - 8}{x^2 - 2x - 8}$, $\{x \mid -2 < x < 4\}$

7. (i) (c) $x + 1$ (d) R
(ii) (c) $x^2 + 2x$ (d) R
(iii) (c) $x^2 + 1$ (d) $\{x \mid 0 \leq x \leq 2\}$
(iv) (c) $x + \sqrt{x}$ (d) $\{x \mid x \geq 0\}$
(v) (c) $x^3 - x$ (d) R
(vi) (c) $\sqrt{16 - x^2} + x$ (d) $\{x \mid -4 \leq x \leq 4\}$

9. $x = 2$ and $x = -3$
10. (a) domain of f $\{x \mid x \neq \pm 2\}$, domain of g $\{x \mid x \neq 2\}$, domain of f + g $\{x \mid x \neq \pm 2\}$
(b) $-\frac{9}{4}$
11. (c) (i) it becomes small (ii) it becomes large

EXERCISE 6.5

1. (a) translate f upward by 6 units
(b) translate f to the left by 6 units
(c) translate f downward by 8 units
(d) translate f to the right by 8 units
(e) stretch f vertically by a factor of 3
(f) reflect f in the x-axis
(g) stretch f vertically by a factor of 5 and reflect in the x-axis
(h) shrink f vertically by a factor of $\frac{1}{5}$
(i) shrink f vertically by a factor of $\frac{1}{3}$ and reflect in the x-axis
(j) translate f to the right by 2 units
(k) translate f upward by 3 units
(l) translate f to the left by 1 unit
(m) stretch f vertically by a factor of 2
(n) stretch f vertically by a factor of 2 and translate upward by 1 unit
(o) translate to the left 1 unit
(p) translate f to the left by 1 unit, then stretch vertically by a factor of 2
(q) stretch f vertically by a factor of 4 and reflect in the x-axis
(r) stretch f vertically by a factor of 4, reflect in the x-axis, and translate downward by 3 units
(s) shrink f vertically by a factor of $\frac{1}{2}$
(t) shrink f vertically by a factor of $\frac{1}{2}$ and translate downward by 1 unit

5. (a) $y = (x + 1)^2 + 1$ (b) $y = (x - 3)^2 - 9$
(c) $y = (x - \frac{1}{2})^2 + \frac{11}{4}$ (d) $y = (x - \frac{5}{2})^2 - 11$
(e) $y = -(x - 4)^2 + 4$ (f) $y = -(x - \frac{3}{2})^2 + \frac{9}{4}$
(g) $y = 2(x + 2)^2 - 11$ (h) $y = 3(x - 1)^2 - 2$
(i) $y = (x + \frac{1}{2})^2 + \frac{3}{4}$ (j) $y = -2(x + 4)^2 + 37$

9. (a) $h(-3) = 0$, $h(-2) = 2$, $h(-1) = 2$, $h(0) = 4$, $h(1) = 4$, $h(2) = 2$, $h(3) = 2$, $h(4) = 0$
11. (a) a reflection in the y-axis
(b) If $-1 < b < 0$, a horizontal stretch by a factor of $\dfrac{1}{b}$ and reflection in the y-axis
If $b < -1$, a horizontal shrink by a factor of $\dfrac{1}{b}$ and reflection in the y-axis
If $b = -1$, a reflection in the y-axis
12. (a) stretch f vertically by a factor of 3 and translate upward 2 units
(b) reflect f in the x-axis and translate upward 1 unit
(c) translate f to the left 5 units
(d) shrink f horizontally by a factor of $\frac{1}{5}$
(e) shrink f horizontally by a factor of $\frac{1}{5}$, then expand vertically by a factor of 3