LESSON PLAN

Course: Grade 12 U Advanced Functions **Lesson: __2 - 8__**

Unit/Chapter: __Functions__ **Topic: _Cubic & Quartics**

- ***homework check:*** collect **Graphing Rational Functions**

- ***note:*** **Graphing Cubic and Quartic Functions**

We know from experience that a cubic equation has three roots. There are several variations as to how those roots occur mixing both real roots with complex (imaginary) roots. The same holds true for quartic equations, however it can have at most four real roots. These roots can be combined into real and complex, equal or distinct.

We apply the same ideas we learned in our investigation of quadratic functions to the graph of associated cubic and quartic functions.

Cubic Functions:

3 real equal roots – touches the x – axis once
3 real distinct roots – crosses the x – axis in three distinct places
1 real and 2 complex roots – crosses the x – axis once with a min or max floating above or below the axis

Quartic Functions:

4 real equal roots – touches x-axis once
4 real distinct roots – crosses axis in four distinct places
2 real equal, 2 real distinct – touches axis once and crosses twice
2 real equal and 2 imaginary – touches once, one max or min floating
2 real distinct and 2 imaginary – crosses twice, one max or min floating
4 imaginary – two max or mins floating, both above or both below

example)

Graph the function $y = x^3 - 3x^2 - 5x + 10$, over the interval $-3 \le x \le 5$

$P(x) = x^3 - 3x^2 - 5x + 10$

$P(-2) = 0$

$$\begin{array}{r|rrrr} -2 & 1 & -3 & -5 & 10 \\ & & -2 & 10 & -10 \\ \hline & 1 & -5 & 5 & 0R \end{array}$$

$y = x^3 - 3x^2 - 5x + 10$

$y = (x+2)(x^2 - 5x + 5)$

$x = -2$

$$x = \frac{5 \pm \sqrt{25 - 4(1)(5)}}{2}$$

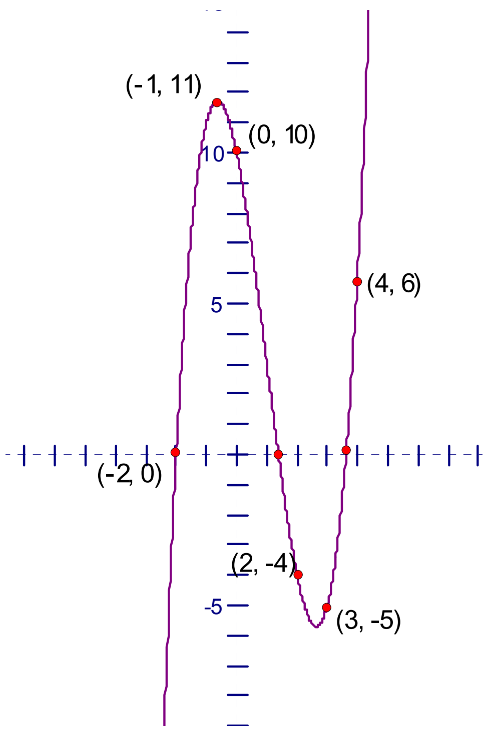
$$x = \frac{5 \pm \sqrt{5}}{2}$$

$$(x \simeq 3.62 \; or \; 1.38)$$

Therefore, the cubic function $y = x^3 - 3x^2 - 5x + 10$ has three real and distinct roots that reflect three x-intercepts. To find out what happens between the intercepts, we can use a table of values that reflects the domain we are given.

x	**y**
-3	**-29**
-2	**0**
-1	**11**
0	**10**
1	**3**
2	**-4**
3	**-5**
4	**6**
5	**35**

The graph is a smooth continuous curve passing through the points indicated in the table.



Note that the cubic curve has one minimum and one maximum point reflected in the “bumps” of the graph. Also note, there are intervals of both positive and negative behaviour. The graph is positive between $-2 < x < 1.38$ and $x > 3.62$. Negative behaviour is reflected in the domain $x < -2$ and $1.38 < x < 3.62$.

Graph the function $f(x) = x^4 - x^3 - 13x^2 + x + 12$ over the interval $-4 \le x \le 5$.

$P(x) = x^4 - x^3 - 13x^2 + x + 12$

Since $P(1) = 0$ and $P(-1) = 0$, we have the factors $(x+1)$ and $(x-1)$

Knowing then that $(x+1)(x-1) = x^2 - 1$, we can divide our polynomial $P(x)$ by the quadratic $x^2 - 1$.

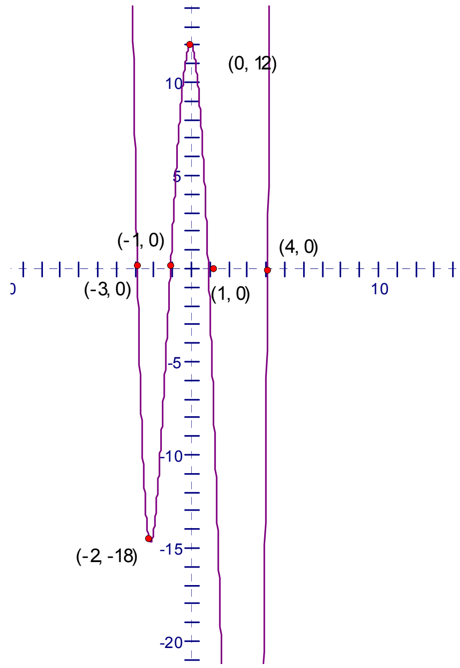
$$
\begin{array}{rl}
 & \qquad\qquad\qquad x^2 - x - 12 \\
x^2 - 1 & \overline{)\,x^4 - x^3 - 13x^2 + x + 12} \\
 & -\left(x^4 \qquad -x^2\right) \\
\cline{2-2}
 & \qquad -x^3 - 12x^2 + x \\
 & \qquad -\left(x^3 \qquad -x\right) \\
\cline{2-2}
 & \qquad\qquad -12x^2 \qquad +12 \\
 & \qquad\quad -\left(-12x \qquad -12\right) \\
\cline{2-2}
 & \qquad\qquad\qquad\qquad\quad 0\ \text{R}
\end{array}
$$

Therefore,
$$
\begin{aligned}
P(x) &= (x+1)(x-1)\left(x^2 - x - 12\right) \\
&= (x+1)(x-1)(x-4)(x+3)
\end{aligned}
$$

If we know that roots are at x= 1, -1, -3, and 4, we can check the values of the remaining points of the domain using a small table of values.

x	**y**
-4	**120**
-2	**-18**
0	**12**
2	**-30**
3	**-48**
5	**192**

The graph will be a smooth continuous curve through the points indicated by the roots and the remaining table points.



Note that the quartic curve has two minimum points and one maximum point reflected by the "bumps" in the graph. Also note the intervals of positive behaviour from $x < -3, \quad -1 < x < 1,$ and $x > 4$ and the intervals of negative behaviour from $-3 < x < -1,$ and $1 < x < 4$.

⌘ ***homework assignment:*** **<u>Gage Mathematics 12</u>** exercise 4.6 p. 181 # 3 – 6

Exercise 4.6

3. Write each of the given polynomials as a product of linear factors, and deduce all their zeros. State the intervals in which the polynomials are positive.

   (a) $x^2 - 9$
   (b) $x^2 + 5x - 6$
   (c) $x^3 - 7x - 6$
   (d) $x^3 + 8x^2 + 7x$
   (e) $x^4 - 16$
   (f) $x^4 + 5x^2 + 4$

4. Find all real and complex roots of the given equations.

   (a) $x^3 - x^2 + 5x = 0$
   (b) $x^3 + 3x = 0$
   (c) $x^3 + 6x^2 + 11x + 6 = 0$
   (d) $2x^3 - 9x^2 = 0$
   (e) $x^4 - 16x^2 = 0$
   (f) $(x^2 + 1)(x^2 + 9) = 0$

5. Sketch graphs of the functions f defined by the following equations in the interval $|x| \leq 4$. In each case use the graph to estimate the real roots of $f(x) = 0$.

   (a) $f(x) = x^2 + x^4$
   (b) $f(x) = x^4 - 2x^2 + 1$
   (c) $f(x) = x^3 - 2x + 1$
   (d) $f(x) = x^3 + x$
   (e) $f(x) = 4x - x^3$
   (f) $f(x) = x^4 + 3x^2 + 2$

6. In the interval $-1 \leq x \leq 1$, sketch graphs of the functions f defined by the given polynomials. Plot values for $x = -1, -\frac{1}{2}, -\frac{1}{4}, 0, \frac{1}{4}, \frac{1}{2}, 1$.

   (a) $f(x) = x^3$
   (b) $f(x) = 2x^3 - x^2$
   (c) $f(x) = x^4$
   (d) $f(x) = x + x^4$

Exercise 4.6

4. (a) $x = 0, \dfrac{1+\sqrt{19}i}{2}, \dfrac{1-\sqrt{19}i}{2}$

(b) $x = 0, i\sqrt{3}, -i\sqrt{3}$

(c) $x = -1, -2, -3$

(d) $x = 0, 0, \dfrac{9}{2}$

(e) $x = 0, 4, -4$

(f) $x = i, -i, 3i, -3i$

5.

(a)

x	y
-4	272
-3	90
-2	20
-1	2
0	0
1	2
2	20
3	90
4	272

The real root is 0.

(b)

x	y
-4	225
-3	64
-2	9
-1	0
0	1
1	0
2	9
3	64
4	225

The real roots are -1, 1.

(c)

x	y
-4	-55
-3	-20
-2	-3
-1	2
0	1
0.5	0.125
0.8	-0.09
1	0
2	5
3	22
4	57

The real roots are 1, -1.62, 0.62.

(d)

x	y
-4	-68
-3	-30
-2	-10
-1	-2
0	0
1	2
2	10
3	30
4	68

0 is the only real root.

(e)

x	y
-4	48
-3	15
-2	0
-1	-3
0	0
1	3
2	0
3	-15
4	-48

-2, 0, 2 are roots.

(f)

x	y
-4	306
-3	110
-2	30
-1	6
0	2
1	6
2	30
3	110
4	306

No real roots.

6.

(a)

x	y
-1	-1
$-\frac{1}{2}$	$-\frac{1}{8}$
$-\frac{1}{4}$	$-\frac{1}{64}$
0	0
$\frac{1}{4}$	$\frac{1}{64}$
$\frac{1}{2}$	$\frac{1}{8}$
1	1

(b)

x	y
-1	-3
$-\frac{1}{2}$	$-\frac{1}{2}$
$-\frac{1}{4}$	$-\frac{3}{32}$
0	0
$\frac{1}{4}$	$-\frac{1}{32}$
$\frac{1}{2}$	0
1	1

(c)

x	y
-1	1
$-\frac{1}{2}$	$\frac{1}{16}$
$-\frac{1}{4}$	$\frac{1}{256}$
0	0
$\frac{1}{4}$	$\frac{1}{256}$
$\frac{1}{2}$	$\frac{1}{16}$
1	1

(d)

x	y
-1	0
$-\frac{1}{2}$	$-\frac{7}{16}$
$-\frac{1}{4}$	$-\frac{63}{256}$
0	0
$\frac{1}{4}$	$\frac{65}{256}$
$\frac{1}{2}$	$\frac{9}{16}$
1	2