LESSON PLAN

Course: Grade 12 U Advanced Functions **Lesson: 6 - 8**

Unit/Chapter: Other Function Types **Topic: The Natural Logarithmic Function**

- ***homework check: :*** **HRW** exercise 4.3 p. 271 # 29 – 40
exercise 4.3A p. 276 – 277

- ***note:*** **The Natural Logarithmic Function**

You know how that basic exponential and logarithmic functions are related through inverse operations. Another basic function similar to the exponential function is known as the natural exponential function with base e which is approximately equal to 2.718…

The log of the natural logarithm e is called the ln (pronounced lawn), therefore the inverse of e^x is $\ln x$. Therefore, the ln button on your calculator is calculated as the log of the natural logarithm with base e. We can use the natural logarithm to help us solve the same types of problems where the bases cannot be made common, keeping in mind, the rules for logs hold true for lns and therefore, $\ln_e e = 1$. For example,
Previously, we might use logs to the base ten to solve,

$5^x = 2$		$5^x = 2$
$\log 5^x = \log 2$		$\ln 5^x = \ln 2$
$x \log 5 = \log 2$	but, we could also use	$x \ln 5 = \ln 2$
$x = \dfrac{\log 2}{\log 5}$		$x = \dfrac{\ln 2}{\ln 5}$
$x \doteq 0.4307$		$x \doteq 0.4307$

As you can see, we get the same answer.

Solve $e^x - e^{-x} = 4$
If we multiple both sides by e^x, we get

$$e^x\left(e^x - e^{-x}\right) = e^x(4)$$
$$e^{2x} - e^x \cdot e^{-x} = 4e^x \quad .$$
$$e^{2x} - 4e^x - 1 = 0$$

Because we cannot factor, let $e^x = u$, and use the quadratic formula on $u^2 - 4u - 1 = 0$ to get

$$u = \frac{-(-4) \pm \sqrt{(-4)^2 - 4(1)(1)}}{2(1)}$$
$$u = \frac{4 \pm \sqrt{20}}{2}$$
$$u = \frac{4 \pm 2\sqrt{5}}{2}$$
$$u = 2 \pm \sqrt{5}$$

Just as we cannot take the log of a negative number, we must also rule the negative answer here as extraneous. Therefore, $u = 2 + \sqrt{5}$ is the only possible answer. If $u = 2 + \sqrt{5}$, then $e^x = 2 + \sqrt{5}$. To solve this equation, we can use the natural logs.

$$e^x = 2 + \sqrt{5}$$
$$\ln e^x = \ln\left(2 + \sqrt{5}\right)$$
$$x \ln e = \ln\left(2 + \sqrt{5}\right)$$
$$x = \ln\left(2 + \sqrt{5}\right)$$
$$x \doteq 1.4436$$

□ ***homework assignment:*** **<u>HRW</u>** exercise 5.6 p. 386 # 1 – 29, 37 – 51 (odds)

Solution

$$\begin{aligned}
\log(x-16) &= 2 - \log(x-1)\\
\log(x-16) + \log(x-1) &= 2\\
\log[(x-16)(x-1)] &= 2\\
\log(x^2 - 17x + 16) &= 2\\
10^{\log(x^2-17x+16)} &= 10^2\\
x^2 - 17x + 16 &= 100\\
x^2 - 17x - 84 &= 0\\
(x+4)(x-21) &= 0\\
x + 4 = 0 \quad \text{or} \quad x - 21 &= 0\\
x = -4 \qquad\quad x &= 21
\end{aligned}$$

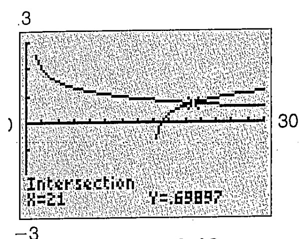


Figure 5.6-12

Because $\log(x-16)$ and $\log(x-1)$ are not defined for $x = -4$, it cannot be a solution. Therefore, the only solution is $x = 21$. The intersection of the graphs of $Y_1 = \log(x-16)$ and $Y_2 = 2 - \log(x-1)$, shown in Figure 5.6-12, confirms the solution. ■

Exercises 5.6

In Exercises 1–8, solve the equation without using logarithms.

1. $3^x = 81$ **2.** $3^x + 3 = 30$ **3.** $3^{x+1} = 9^{5x}$

4. $4^{5x} = 16^{2x-1}$ **5.** $3^{5x}9^{x^2} = 27$ **6.** $2^{x^2+5x} = \frac{1}{16}$

7. $9^{x^2} = 3^{-5x-2}$ **8.** $4^{x^2-1} = 8^x$

In Exercises 9–29, solve the equation. Give exact answers (in terms of natural logarithms). Then use a calculator to find an approximate answer.

9. $3^x = 5$ **10.** $5^x = 4$

11. $2^x = 3^{x-1}$ **12.** $4^{x+2} = 2^{x-1}$

13. $3^{1-2x} = 5^{x+5}$ **14.** $4^{3x-1} = 3^{x-2}$

$2^{1-3x} = 3^{x+1}$ **16.** $3^{z+3} = 2^z$

17. $e^{2x} = 5$ **18.** $e^{-3x} = 2$

19. $6e^{-1.4x} = 21$ **20.** $3.4e^{-\frac{x}{3}} = 5.6$

21. $2.1e^{\frac{x}{2}\ln 3} = 5$ **22.** $7.8e^{\frac{x}{3}\ln 5} = 14$

24. $4^x - 6 \cdot 2^x = -8$

25. $e^{2x} - 5e^x + 6 = 0$ *Hint:* Let $u = e^x$.

26. $2e^{2x} - 9e^x + 4 = 0$ **27.** $6e^{2x} - 16e^x = 6$

28. $8e^{2x} + 8e^x = 6$ **29.** $4^x + 6 \cdot 4^{-x} = 5$

In Exercises 30–32, solve the equation for x.

30. $\frac{e^x + e^{-x}}{e^x - e^{-x}} = t$ **31.** $\frac{e^x - e^{-x}}{2} = t$ **32.** $\frac{e^x - e^{-x}}{e^x + e^{-x}} = t$

33. Prove that if $\ln u = \ln v$, then $u = v$. *Hint:* Use the basic property of inverses $e^{\ln v} = v$.

34. **a.** Solve $7^x = 3$ using natural logarithms. Give an exact answer, not an approximation.

b. Solve $7^x = 3$ using common logarithms. Give an exact answer, not an approximation.

c. Use the change-of-base formula in *Excursion* 5.5.A to show that your answers in parts **a** and **b** are the same.

In Exercises 35–44, solve the equation. (See Example 9.)

37. $\log(3x - 1) + \log 2 = \log 4 + \log(x + 2)$

38. $\ln(x + 6) - \ln 10 = \ln(x - 1) - \ln 2$

39. $2 \ln x = \ln 36$

40. $2 \log x = 3 \log 4$

41. $\ln x + \ln(x + 1) = \ln 3 + \ln 4$

42. $\ln(6x - 1) + \ln x = \frac{1}{2} \ln 4$

43. $\ln x = \ln 3 - \ln(x + 5)$

44. $\ln(2x + 3) + \ln x = \ln e$

In Exercises 45–52, solve the equation.

45. $\ln(x + 9) - \ln x = 1$

46. $\ln(2x + 1) - 1 = \ln(x - 2)$

47. $\log x + \log(x - 3) = 1$

48. $\log(x - 1) + \log(x + 2) = 1$

49. $\log \sqrt{x^2 - 1} = 2$

50. $\log \sqrt[3]{x^2 + 21x} = \frac{2}{3}$

51. $\ln(x^2 + 1) - \ln(x - 1) = 1 + \ln(x + 1)$

52. $\frac{\ln(x + 1)}{\ln(x - 1)} = 2$

Exercises 53–62 deal with the half-life function $M(x) = c(0.5)^{\frac{x}{h}}$, which was discussed in Section 5.3 and used in Example 5 of this section.

53. How old is a piece of ivory that has lost 36% of its carbon-14?

54. How old is a mummy that has lost 49% of its carbon-14?

55. Find when part of the Pueblo Benito ruins was built if the doorway timbers have 89.14% of their original carbon-14. (See the image on the first page of this chapter.)

56. How old is a wooden statue that has only one-third of its original carbon-14?

57. A quantity of uranium decays to two-thirds of its original mass in 0.26 billion years. Find the half-life of uranium.

58. A certain radioactive substance loses one-third of its original mass in 5 days. Find its half-life.

59. Krypton-85 loses 6.44% of its mass each year. What is its half-life?

60. Strontium-90 loses 2.5% of its mass each year. What is its half-life?

61. The half-life of a certain substance is 3.6 days. How long will it take for 20 grams to decay to 3 grams?

62. The half-life of cobalt-60 is 4.945 years. How long will it take for 25 grams to decay to 15 grams?

Exercises 63–68 deal with the compound interest formula $A = P(1 + r)^t$, which was discussed in Section 5.3 and used in Example 6 of this section.

63. At what annual rate of interest should $1000 be invested so that it will double in 10 years, if interest is compounded quarterly?

64. Find how long it takes $500 to triple if it is invested at 6% in each compounding period.
a. annually **b.** quarterly **c.** daily

65. **a.** How long will it take to triple your money if you invest $500 at a rate of 5% per year compounded annually?
b. How long will it take at 5% compounded quarterly?

66. At what rate of interest compounded annually should you invest $500 if you want to have $1500 in 12 years?

67. How much money should be invested at 5% interest compounded quarterly so that 9 years later the investment will be worth $5000? This answer is called the present value of $5000 at 5% interest.

68. Find a formula that gives the time needed for an investment of P dollars to double, if the interest rate is r% compounded annually. *Hint:* Solve the compound interest formula for t, when $A = 2P$.

Exercises 69–76 deal with functions of the form $f(x) = Pe^{kx}$, where k is the continuous exponential growth rate. See Example 7.

69. The present concentration of carbon dioxide in the atmosphere is 364 parts per million (ppm) and is increasing exponentially at a continuous yearly rate of 0.4% (that is, $k = 0.004$). How many years will it take for the concentration to reach 500 ppm?

70. The amount P of ozone in the atmosphere is currently decaying exponentially each year at a

33. False; the graph of the left side differs from the graph of the right side.

35. Answers may vary: $\frac{\log 3}{\log 2} = 1.585$ and $\log\left(\frac{3}{2}\right) = 0.1761$ thus $\frac{\log 3}{\log 2} \neq \log\left(\frac{3}{2}\right)$

37. $b = e$ **39.** A = 3, B = 2 **41.** 2

43. Approximately 2.54 **45.** 20 decibels

47. Approximately 66 decibels **49.** 100 times

51. a. 1.2553 **b.** 3.9518 **c.** $\log x = \frac{\ln x}{\ln 10}$

Section 5.5.A, page 376

1. $\log 0.01 = -2$ **3.** $\log \sqrt[3]{10} = \frac{1}{3}$

5. $\log r = 7k$ **7.** $\log_7 5{,}764{,}801 = 8$

9. $\log_3\left(\frac{1}{9}\right) = -2$ **11.** $10^4 = 10{,}000$

13. $10^{2.8751} \approx 750$ **15.** $5^3 = 125$ **17.** $2^{-2} = \frac{1}{4}$

19. $10^{z+w} = x^2 + 2y$ **21.** $\sqrt{43}$ **23.** $\sqrt{x^2 + y^2}$

25. $\frac{1}{2}$ **27.** 6

29.

x	0	1	2	4
$f(x) = \log_4 x$	Not defined	0	0.5	1

31.

x	$\frac{1}{36}$	$\frac{1}{6}$	1	216
$h(x) = \log_6 x$	−2	−1	0	3

33.

x	0	$\frac{1}{7}$	$\sqrt{7}$	49
$f(x) = 2\log_7 x$	Not defined	−2	1	4

35.

x	−2.75	−1	1	29
$h(x) = 3\log_2(x + 3)$	−6	3	6	15

37. $b = 3$ **39.** $b = 20$

41. 5 **43.** 3 **45.** 4 **47.** $\log \frac{x^2y^3}{z^6}$

49. $\log(x^2 - 3x)$ **51.** $\log_2(5c)$ **53.** $\log_4\left(\frac{1}{49c^2}\right)$

55. $\ln\left(\frac{(x+1)^2}{x+2}\right)$ **57.** $\log_2(x)$ **59.** $\ln(e^2 - 2e + 1)$

61. 3.3219 **63.** 0.8271 **65.** 1.1115 **67.** 1.6199

Domain: all real numbers $> \frac{4}{3}$

Range: all real numbers

71. Compress the graph vertically by a factor of $\frac{1}{3}$, then a horizontal translation of 1 unit to the right, then a vertical translation of 7 units upward.
Domain: all real numbers > 1
Range: all real numbers

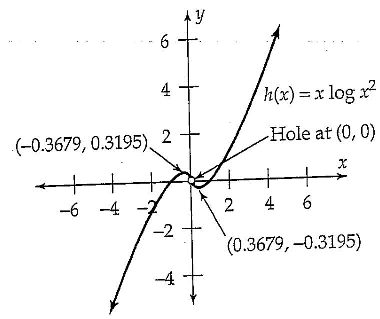
73. True **75.** True **77.** False **79.** 397^{398}

81. $\log_b u = \frac{\log_a u}{\log_a b}$ **83.** $\log_{10} u = 2\log_{100} u$

85. $\log_b x = \frac{1}{2}\log_b v + 3 = \log_b \sqrt{v} + \log_b b^3 = \log_b(b^3 \cdot \sqrt{v})$; hence $x = b^3\sqrt{v}$.

87. $f(x) = g(x)$ only when $x \approx 0.123$, so the statement is false.

89.



Section 5.6, page 386

1. $x = 4$ **3.** $x = \frac{1}{9}$ **5.** $x = \frac{1}{2}$ or -3

7. $x = -2$ or $-\frac{1}{2}$ **9.** $x = \frac{\ln 5}{\ln 3} \approx 1.465$

11. $x = \frac{\ln 3}{\ln 1.5} \approx 2.7095$

13. $x = \frac{\ln 3 - 5\ln 5}{\ln 5 + 2\ln 3} \approx -1.825$

15. $x = \frac{\ln 2 - \ln 3}{3\ln 2 + \ln 3} \approx -0.1276$

17. $x = \frac{(\ln 5)}{2} \approx 0.805$ **19.** $x = \frac{(-\ln 3.5)}{1.4} \approx -0.895$

21. $x = \frac{2\ln\left(\frac{5}{2.1}\right)}{\ln 3} \approx 1.579$ **23.** $x = 0$ or 1

25. $x = \ln 2 \approx 0.693$ or $x = \ln 3 \approx 1.099$

7. $x = \ln 3 \approx 1.099$

9. $x = \dfrac{\ln 2}{\ln 4} = \dfrac{1}{2}$ or $x = \dfrac{\ln 3}{\ln 4} \approx 0.792$

1. $x = \ln\left(t + \sqrt{t^2 + 1}\right)$

3. If $\ln u = \ln v$, then $e^{\ln u} = e^{\ln v}$, so $u = v$

5. $x = 9$ **37.** $x = 5$ **39.** $x = 6$ **41.** $x = 3$

3. $x = \dfrac{-5 + \sqrt{37}}{2}$ **45.** $x = \dfrac{9}{(e-1)}$ **47.** $x = 5$

9. $x = \pm\sqrt{10001}$ **51.** $x = \sqrt{\dfrac{e+1}{e-1}}$

3. Approximately 3689 years old

5. Approximately 950.35 years ago

7. Approximately 444,000,000 years

9. Approximately 10.413 years

1. Approximately 9.853 days

3. Approximately 6.99%

5. **a.** Approximately 22.5 years
   b. Approximately 22.1 years

7. \$3197.05 **69.** 79.36 years

1. **a.** About 2.1548% **b.** In the year 2013

a. $k \approx 21.459$ **b.** $t \approx 0.182$

5. **a.** There are 20 bacteria at the beginning and 2500 three hours later.
   b. $\dfrac{\ln 2}{\ln 5} \approx 0.43$

7. **a.** At the outbreak: 200 people; after 3 weeks: about 2717 people
   b. In about 6.09 weeks

79. **a.** $k \approx 0.229, c \approx 83.3$
   b. 12.43 weeks

Section 5.7, page 396

1. Cubic, exponential, logistic

3. Exponential, quadratic, cubic

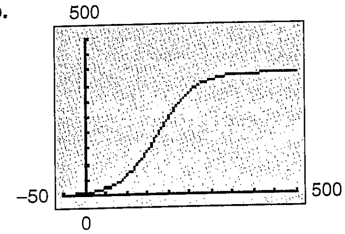
5. Exponential, logarithmic, quadratic, cubic

7. Quadratic, cubic **9.** Quadratic, cubic

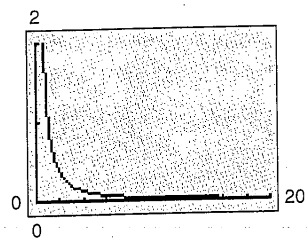
11. Ratios: 5.07, 5.06, 5.06, 5.08, 5.05; exponential is appropriate

13. **a.** For large values of x the term $54e^{-0.0228x}$ is close to zero so the quantity $(1 + 54e^{-0.0228x})$ is slightly larger than 1, which means $\dfrac{384.57}{1 + 54e^{-0.0228x}}$ is always less than (but very close to) 384.57.

b.



15. a.



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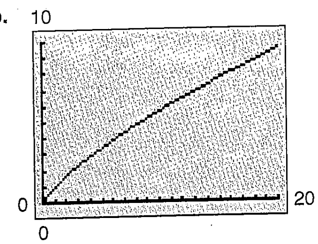
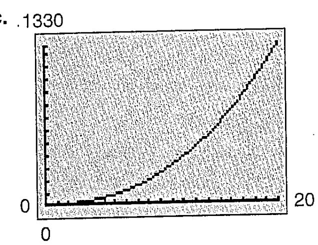


c.



17. $\{(\ln x, \ln y)\}$ appears the most linear. Power model

19. $\{(\ln x, \ln y)\}$ and $\{(\ln x, y)\}$ are both nearly linear. Power or logarithmic model

21. a.